The Mixmaster Model as a Cosmological Framework and Aspects of its Quantum Dynamics

Giovanni Imponente [1,2] * and Giovanni Montani [1,3] †

[1] ICRA—International Center for Relativistic Astrophysics, Piazza A.Moro 5 00185 Rome, Italy
[2] Università degli Studi di Napoli "Federico II" and INFN—Sezione di Napoli
[3] Physics Department, University of Rome "La Sapienza"

Abstract: This paper provides a review of some recent issues on the Mixmaster dynamics concerning the features of its stochasticity. After a description of the geometrical structure characterizing the homogeneous cosmological models in the Bianchi classification and the Belinsky-Khalatnikov-Lifshitz piecewise representation of the types VIII and IX oscillatory regime, we face the question regarding the time covariance of the resulting chaos as viewed in terms of continuous Misner-Chitré like variables. Finally we show how in the statistical mechanics framework the Mixmaster chaos raises as semiclassical limit of the quantum dynamics in the Planckian era.

PACS number(s): 04.20.Jb, 98.80.Dr

I. INTRODUCTION

As well-known the Standard Cosmological Model (SCM) finds its theoretical basis in terms of a homogeneous and isotropic universe obtained as high symmetry solution of the Einstein's equations, the so called Friedmann-Robertson-Walker model (FRW). Such representation of our actual universe possesses a clear degree of reliability due to its good general agreement with respect to the observed phenomenology (in particular the strong isotropy of the Cosmic Background Radiation as well as the consistency between the predictions on the primordial nucleosynthesis of the light elements and the experimentally observed abundances), nevertheless there are some important general aspects to be taken into account.

In first place none theoretical principle led us to exclude that in the very early phases of its evolution the universe had been characterized by a higher degree of inhomogeneity and anisotropy and only in a later stage underwent an isotropization process as natural consequence of its dynamics and/or by the action of some physical mechanism in classical as well as quantum regime (to which the causality notion plays a crucial role); indeed the instability of the FRW solution toward the cosmological singularity implies that a more general behavior should characterize the very early evolution as soon as we make allowance for small perturbations responsible for the actual clumpyness.

Furthermore the SCM in its original formulation contains non-trivial internal inconsistencies which require to be explained through appropriate modifications of the underlying theory. Among this undesired theoretical facts take particular importance the so called "horizon paradox" and "flatness problem" to which should be added the absence of an appropriate model for the large-scale structures formation able to reproduce the observed distribution of matter in the actual universe [51,50,52].

All these considerations make clear the deep interest in studying more general classes of solutions of the Einstein's equations in order to individualize dynamical behaviors which could constitute a more suitable framework than the simple FRW model for the construction of a completely self-consistent cosmology, with particular reference to the universe evolution in proximity of the initial "Big Bang" [54,34,53].

In this paper (Sections I and II) we will discuss one of the most studied cosmological dynamical models in view of its possible implications in the history of our universe, the so called Mixmaster model [6] [7,19]. It corresponds to the asymptotic evolution toward the cosmological singularity of the type models VIII and IX in the famous Bianchi classification [15]. Another important theoretical property characterizing such model is that it is an appropriate prototype of the behavior of the general cosmological solution of the Einstein's equations in the same asymptotic region to the initial singularity. Indeed the direct extension to the inhomogeneous case of the ideas and the formalism required by the treatment of the homogeneous one, extension based on the dynamical decoupling characterizing near the singularity the different points of the space, leads to derive in natural way the asymptotic evolution to the "Big Bang" of a generic inhomogeneous cosmological model which constitutes surely one of the most important results until now obtained in Relativistic Cosmology [7,19].

The description of the Mixmaster Model can take place in terms of two different, but indeed completely equivalent approaches: one is based on the direct analysis of the Einstein's equations, as in the original analysis due to Belinski, Khalatnikov and Lifshitz (BKL) (Section III), while the other corresponds to a Hamiltonian approach to the dynamics (Section IV) which had been introduced first by Misner [6].

*E-mail address: imponente@icra.it
†E-mail address: montani@icra.it

arXiv:gr-qc/0201102v1 31 Jan 2002

However not all anisotropic dynamics are compatible with a satisfactory SCM but, as shown in the early Seventies, under suitable conditions some can be represented as a FRW model plus a gravitational waves packet ([12], [13]).

Among the Bianchi classification, the types VIII[1] and IX appear as the most general ones: the former's geometry is invariant under the $SO(3)$ group, like the closed FRW universe and its dynamics allows the line element to be decomposed as

$$ds^2 = ds_0{}^2 - \delta_{(a)(b)} G_{ik}^{(a)(b)} dx^i dx^k \qquad (1)$$

where ds_0 denotes the line element of an isotropic universe having positive constant curvature, $G_{ik}^{(a)(b)}$ is a set of spatial tensors[2] and $\delta_{(a)(b)}(t)$ are amplitude functions, resulting small sufficiently far from the singularity.

Since Belinski, Khalatnikov and Lifshitz [7] derived the oscillatory regime characterizing the evolution near a physical singularity, a wide literature faced over the years this subject in order to understand the resulting chaos.

The research developed overall in two different, but related, directions: firstly, the dynamical analysis removed the limits of the BKL approach due to its discrete nature, providing satisfactory representations in terms of continuous variables (i.e. construction of an invariant measure for the system [21], [48]); secondly, a better characterization of the Mixmaster chaos (in view of its properties of covariance) found non-trivial difficulties to apply the standard chaos indicators to relativistic systems.

The various approaches prevented, up to now, to say a definitive word about the covariance of Mixmaster chaos, though important indications in favor of such covariance arise from [47], [60], [57] and [58,59], as above discussed in detail.

A wide interest in understanding this intrinsic nature of the chaotic Mixmaster dynamics and its very early appearance in the Universe evolution, to which we dedicate our analysis, lead to believe in the existence of a relation with the quantum behavior the system performs during the Planckian era. In fact the last aim of this paper (Section V) is to give a precise meaning to such relation by constructing the semiclassical limit of a Schöedinger approach to the canonical quantization of the Arnowitt-Deser-Misner (ADM) dynamics.

---

[1]All the considerations we will develop for the type IX apply also to the VIII one since, close to the singularity, they have the same morphology.

[2]These tensors satisfy the equations

$$G_{ik\ ;l}^{(a)(b);l} = -(n^2-3) G_{ik}^{(a)(b)}, \quad G_{;k}^{(a)(b)k} = 0, \quad G_i^{(a)(b)i} = 0,$$

in which the Laplacian is referred to the geometry of the sphere of unit radius.

II. GEOMETRICAL STRUCTURE OF THE BIANCHI MODELS

A. Group Transformations

A (pseudo) Riemannian space, given a metric g and a differential structure (M, g), is homogeneous if it is orbit for an isometry group as a group invariance for the metric g. The groups of movements for the metric are said to be isometry groups and for homogeneous spaces they are transitive, in the sense that each point is equivalent under the action of the group and they are spatial sections of the space-time.

We are interested in the Lie algebras for Killing vector fields as generators of these groups of movements, intended as infinitesimal transformations generating rotations and translations.

Let's consider a group of transformations

$$x^\mu \rightarrow \bar{x}^\mu = f^\mu(x, a) \qquad (2)$$

over a space M, where $\{a^a\}_{a=1,..,r}$ are r independent variables parameterizing the group and a_0 corresponds to the identical transformation

$$f^\mu(x, a_0) = x^\mu. \qquad (3)$$

Let's then consider also the infinitesimal transformation corresponding to $a_0 + \delta a$, close to the identity,

$$x^\mu \rightarrow \bar{x}^\mu = f^\mu(x, a_0 + \delta a) \approx$$
$$\approx \underbrace{f^\mu(x, a_0)}_{=x^\mu} + \underbrace{\left(\frac{\partial f^\mu}{\partial a^a}\right)(x, a_0)}_{\equiv \xi_a^\mu(x)} \delta a^a \qquad (4)$$

say

$$x^\mu \rightarrow \bar{x}^\mu \approx x^\mu + \xi_a^\mu(x) \delta a^a = (1 + \delta a^a \xi_a) x^\mu, \qquad (5)$$

where the r first order differential operators $\{\xi_a\}$ are defined as $\xi_a = \xi_a^\mu \frac{\partial}{\partial x^\mu}$ in correspondence with the r vectorial fields with components $\{\xi_a^\mu\}$. These are the Killing generating vector fields.

The coordinate transformation can be definitely written as

$$\bar{x}^\mu \approx (1 + \delta a^a \xi_a) \approx e^{\delta a^a \xi_a} x^\mu \qquad (6)$$

and in a finite form

$$\bar{x}^\mu \rightarrow \bar{x}^\mu = e^{\theta^a \xi_a} x^\mu \qquad (7)$$

where $\{\theta^a\}$ are r new parameters for the group.

The vectorial generator fields form a Lie algebra, say a real r-dimensional vectorial space with basis $\{\xi_a\}$, closed with respect to commutation, in order to have the representation

$$[\xi_a, \xi_b] \equiv \xi_a \xi_b - \xi_b \xi_a = C_{ab}^c \xi_c \qquad (8)$$

where C^c_{ab} are the structure constants for the Lie algebra. It is then natural to extend this formalism to the choice of $\{e_a\}$ as basis for the Lie algebra g over a group G with

$$[e_a, e_b] = C^c_{ab} e_c ; \qquad (9)$$

then let's define the symmetric quantity

$$\gamma_{ab} = C^c_{ad} C^d_{bc} = \gamma_{ba} \qquad (10)$$

and an internal product over the Lie algebra

$$\gamma_{ab} \equiv e_a \cdot e_b = \gamma(e_a, e_b), \quad \gamma(X^a e_a, Y^b e_b) = \gamma_{ab} X^a Y^b. \qquad (11)$$

The orbit of x is

$$f_G(x) = \{f_a(x) \mid a \in G\} \qquad (12)$$

as the set of all points that can be achieved from x under the group of transformations. The isotropy group in x is

$$G_x = \{a \in G \mid f_a(x) = x\}, \qquad (13)$$

i.e. the subgroup of G which leaves x fixed. If x_1, x_2 are on the same orbit, then G_{x_1} and G_{x_2} are conjugate subgroups of G and then isomorphic. If now $G_x = \{a_0\}$ and $f_G(x) = M$, G is diffeomorphic to M and the two spaces are identified. If g is a metric over M invariant under G, then it is definitively given by the inner product of the invariant vectorial fields of basis e_a.

The groups of non-Abelian transformations, as defined by (9), represent geometrically homogeneous spaces in three dimensions and give the section spatially homogeneous of the spatially homogeneous space-times.

Given a basis $\{e_a\}$ of the Lie algebra for the tridimensional Lie group, with structure constants C^c_{ab}, at any time the spatial metric is given by the spatially constant inner products

$$e_a \cdot e_b = g_{ab}(t) \qquad a, b = 1, \ldots, 3 \qquad (14)$$

which are six functions of the time variable only. This permits to rewrite the Einstein equations as ordinary differential equations, eventually associated to constraints functions of t, necessary to describe the matter behavior of the universe.

In four dimensions one obtains homogeneous space-times: given a set of structure constants $C^\alpha_{\beta\gamma}$ over a basis $\{e_\alpha\}$ of the four dimensional Lie algebra, the ten constants

$$g_{\alpha\beta} = e_\alpha e_\beta \qquad \alpha, \beta = 1, \ldots, 4 \qquad (15)$$

determine univocally the signature of the Lorentz metric. As a consequence, Einstein's equations reduce to a system of algebraic equations for $g_{\alpha\beta}$ and $C^\alpha_{\beta\gamma}$, which could in principle not be solvable for each transformations' group, in view of the fact that for general formulations with matter fields there are more constants to take account of.

In any case one needs to consider only one representative group for each class of equivalence of the Lie groups. Referring to three dimensions (only the spatial sector), the Bianchi classification [1] determines definitely all symmetries for tridimensional homogeneous spaces, analogously to the curvature ($k = -1, 0, +1$) which distinguishes homogeneous and isotropic spaces (FRW).

B. Einstein's Equations in the Synchronous Gauge

Let's us specify the general scheme outlined above in the specific approach of Luigi Bianchi in 1897 [1] and independently applied to cosmology by Belinski, Khalatnikov and Lifshitz in 1969 [7].

In the cosmological approach we consider a synchronous reference system, so that there is an unique time t synchronized all over the space. This can be obtained requiring the metric tensor g_{ab} to have the peculiar diagonal form such as

$$g_{00} = 1 \quad g_{0\alpha} = 0, \qquad \alpha = 1, \ldots, 3 \qquad (16)$$

in order to write the line element

$$ds^2 = N^2(t) dt^2 - dl^2, \qquad (17)$$

where $N(t)$ is the lapse function (in this section we adopt the synchronous gauge $N \equiv 1$). Writing[3]

$$dl^2 = \gamma_{\alpha\beta}(t, x^\gamma) dx^\alpha dx^\beta, \qquad (18)$$

t is the synchronous time and the tridimensional tensor $\gamma_{\alpha\beta}$ describes the metric spatial sector. The identity $g_{0\alpha} \equiv 0$ ensures the complete equivalence in the three spatial directions, while the unitarity of g_{00} is always valid, provided an appropriate rescaling of the time coordinate. In this reference system the time lines ($x^\gamma = $ const.) are the geodesics of the four dimensional space-time

$$\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = \Gamma^i_{00} = 0, \quad (i, k, l = 0, \ldots, 3) \qquad (19)$$

being $u^i = \frac{dx^i}{ds}$ the tangent four vector to the world line $x^\gamma = $ const. with components $u^0 = 1$, $u^\alpha = 0$ and orthogonal to the hypersurfaces $t = $ const.[4].

---

[3]All over this section Latin indexes run over $0, \ldots, 3$ while Greek indexes $1, \ldots, 3$.

[4]This geometrical construction is always available but not univocally defined: a coordinate transformation of the type

$$\begin{cases} t' & = t \\ x^{\alpha\prime} & = x^{\alpha\prime}(x^\beta) \end{cases} \qquad (20)$$

doesn't affect the time coordinate and defines the passage from a synchronous reference system to another one, fully equivalent.

Under this choice and the metric (17), the mixed components of the Einstein's equations write

$$R_0^0 = -\frac{1}{2}\frac{\partial}{\partial t}\kappa_\alpha^\alpha - \frac{1}{4}\kappa_\alpha^\beta\kappa_\beta^\alpha = 8\pi G\left(T_0^0 - \frac{1}{2}T\right) \qquad (21)$$

$$R_\alpha^0 = \frac{1}{2}\left(\kappa_{\alpha;\beta}^\beta - \kappa_{\beta;\alpha}^\beta\right) = 8\pi G T_\alpha^0 \qquad (22)$$

$$R_\alpha^\beta = -P_\alpha^\beta - \frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\kappa_\alpha^\beta\right) = 8\pi G\left(T_\alpha^\beta - \frac{1}{2}\delta_\beta^\alpha T\right) \quad (23)$$

being

$$\kappa_{\alpha\beta} = \frac{\partial\gamma_{\alpha\beta}}{\partial t}, \qquad \gamma \equiv |\gamma_{\alpha\beta}|, \qquad (24)$$

$T_{\alpha\beta}$ the momentum energy tensor for the system and $P_{\alpha\beta}$ the tridimensional Ricci tensor obtained from the metric $\gamma_{\alpha\beta}$[5].
In (23) the spatial and temporal derivatives are split, while from (21) one obtains the property for the metric determinant to be zero for a certain time (Landau-Raichoudhury theorem)[6], nevertheless the singularity with respect to the temporal parameter is a physical one.

C. Tetradic Representation of the Metric Tensor

The choice of a tetradic basis of four linearly independent vector fields (depending from the system symmetries) permits to project all quantities in a very useful way to obtain new simpler equations to be satisfied.
Consider on each point of the space-time the basis of four linearly independent contravariant vectors $e_{(a)}^i$, $(a = 1, \ldots, 4)$, where the bracketed index is the tetradic one, while i is tensorial, and the covariant set $e_{(a)\ i} = g_{ik}e_{(a)}^k$, being g_{ik} the metric tensor. Define also the reciprocal set $e_i^{(a)}e_{(b)}^i = \delta_b^a$ (orthogonality condition). It is easy to check the validity of

$$e_i^{(a)}e_{(a)}^k = \delta_i^k. \qquad (25)$$

The definition is complete with

$$e_{(a)}^i e_{(b)\ i} = \eta_{ab}, \qquad (26)$$

where η_{ab} is a symmetric matrix with signature $(+ - - -)$ (with the orthonormality of (25)); the corresponding

---

[5]The indexes of these two tensors are raised and lowered using such reduced metric.
[6]This fictitious singularity disappears with a change of reference system and is related to the intersection of the geodesics belonging to any arbitrary set with the envelope surfaces, say to the procedure used to build a synchronous reference system.

inverse matrix is η^{ab}, being $\eta^{ac}\eta_{cb} = \delta_b^a$. Then follow the properties

$$\eta_{(a)(b)}e_i^{(a)} = e_{(b)i} \qquad (27)$$

$$\eta^{(a)(b)}e_{(a)i} = e_i^{(b)} \qquad (28)$$

$$e_{(a)i}e_j^{(a)} = g_{ij}. \qquad (29)$$

For a generic vector (A_j) or tensorial (T_{ij}) field the tetradic components are, in general

$$A_{(a)} = e_{(a)j}A^j = e_{(a)}^j A_j, \qquad (30)$$

$$A^{(a)} = \eta^{(a)(b)}A_{(b)} = e_j^{(a)}A^j = e^{(a)j}A_j, \qquad (31)$$

$$A^i = e_{(a)}^i A^{(a)} = e^{(a)i}A_{(a)} \qquad (32)$$

and, respectively,

$$T_{(a)(b)} = e_{(a)}^i e_{(b)}^j T_{ij} = e_{(a)}^i T_{i(b)}, \qquad (33)$$

$$T_{ij} = e_i^{(a)}e_j^{(b)}T_{(a)(b)} = e_i^{(a)}T_{(a)j}. \qquad (34)$$

The tetradic indexes can be raised or lowered with the use of the tensors $\eta_{(a)(b)}$ and $\eta^{(a)(b)}$, while the contraction gives then a result independent of the indexes nature.
By (26) and (27) one obtains also

$$g_{ik} = e_{(a)i}e_k^{(a)} = \eta_{ab}e_i^{(a)}e_k^{(b)} \qquad (35)$$

so that the line element becomes

$$ds^2 = \eta_{ab}\left(e_i^{(a)}dx^i\right)\left(e_k^{(b)}dx^k\right). \qquad (36)$$

The choice of η_{ab} permits to split the tetradic basis in one temporal and three spatial vectors. Nevertheless the expressions $dx^{(a)} = e_i^{(a)}dx^i$ are not, in general, exact differentials of functions of the coordinates.

D. Directional Derivative

The contravariant set $e_{(a)}$ of tangent vectors leads to the natural covariant derivative definition

$$e_{(a)} = e_{(a)}^i \frac{\partial}{\partial x^i} \qquad (37)$$

so that the derivative of a generic scalar field ϕ along the direction a is

$$\phi_{,(a)} = e_{(a)}^i \frac{\partial\phi}{\partial x^i} = e_{(a)}^i \phi_{,i}. \qquad (38)$$

The general extension of such definition is

$$A_{(a),(b)} = e_{(b)}^i \frac{\partial}{\partial x^i}A_{(a)} = e_{(b)}^i \frac{\partial}{\partial x^i}e_{(a)}^j A_j =$$
$$= e_{(b)}^i \nabla_{\partial_i}\left[e_{(a)}^j A_j\right] = e_{(b)}^i \left[e_{(a)}^j A_{j;i} + A_k e_{(a);i}^k\right] \quad (39)$$

in order to rewrite

$$A_{(a),(b)} = e^j_{(a)} A_{j;i} e^i_{(b)} + e_{(a)k;i} e^i_{(b)} e^k_{(c)} A^{(c)} . \qquad (40)$$

Let's introduce the Ricci's rotation coefficients γ_{abc}

$$\gamma_{abc} = e_{(a)i;k} e^i_{(b)} e^k_{(c)} \qquad (41)$$

and their linear combinations

$$\lambda_{abc} = \gamma_{abc} - \gamma_{acb} = \left(e_{(a)i;k} - e_{(a)k;i} \right) e^i_{(b)} e^k_{(c)} =$$
$$= \left(e_{(a)i,k} - e_{(a)k,i} \right) e^i_{(b)} e^k_{(c)} \qquad (42)$$

in which is has been used the identity

$$A_{i;k} - A_{k;i} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i} . \qquad (43)$$

Expression (42), in which the regular derivatives are substituted by the covariant ones, is invertible

$$\gamma_{abc} = \frac{1}{2} \left(\lambda_{abc} + \lambda_{bca} - \lambda_{cab} \right) . \qquad (44)$$

From the identity

$$0 = \eta_{(a)(b),i} = \left[e_{(a)k} e^k_{(b)} \right]_{;i} \qquad (45)$$

one gets the symmetry properties

$$\gamma_{abc} = -\gamma_{bac} , \qquad \lambda_{abc} = -\lambda_{acb} . \qquad (46)$$

Now the formalism is ready to find the values of the structure constants which leave the metric invariant under the homogeneity constraint.
The basis $e_{(a)}$ permits to express the Lie parentheses as

$$\left[e_{(a)}, e_{(b)} \right] = C^{(c)}_{\ (a)(b)} e_{(c)} \qquad (47)$$

where the coefficients $C^{(c)}_{\ (a)(b)}$ are the 24 (in four dimensional space) structure constants for the group of transformations, antisymmetric with respect to the lower indexes; it is easy to obtain the explicit relation

$$C^{(c)}_{\ (a)(b)} = \gamma^{(c)}_{\ (b)(a)} - \gamma^{(c)}_{\ (a)(b)} . \qquad (48)$$

E. Ricci and Bianchi Identities

By the Riemann tensor $R^m_{\ ikl}$, the Bianchi identity becomes

$$A_{i;k;l} - A_{i;l;k} = A_m R^m_{\ ikl} \qquad (49)$$

for a generic four vector A_i and, applied to the tetradic basis,

$$e_{(a)i;k;l} - e_{(a)i;l;k} = e^m_{(a)} R_{mikl} . \qquad (50)$$

Projecting this expression on the basis itself one obtains

$$R_{(a)(b)(c)(d)} = R_{mikl} e^m_{(a)} e^i_{(b)} e^k_{(c)} e^l_{(d)} =$$
$$= -\gamma_{(a)(b)(c),(d)} + \gamma_{(a)(b)(d),(c)} +$$
$$+\gamma_{(a)(b)(f)} \left[\gamma_{(c)}^{\ (f)}_{\ (d)} - \gamma_{(d)}^{\ (f)}_{\ (c)} \right] +$$
$$+\gamma_{(f)(a)(c)} \gamma_{(b)}^{\ (f)}_{\ (d)} +$$
$$-\gamma_{(f)(a)(d)} \gamma_{(b)}^{\ (f)}_{\ (c)} . \qquad (51)$$

In view of the re-expression of the interesting quantities in terms of the Ricci's rotation coefficients $\gamma_{(a)(b)(c)}$ and subsequently in terms of the structure constants, it is easy to see the great simplification in the formulas, provided a space for which exists a set of such constants $C^{(a)}_{\ (b)(c)}$, like the analysis made by Bianchi for the homogeneous spaces leads to [1].

The length element before the transformation is

$$dl^2 = \gamma_{\alpha\beta} \left(x^1, x^2, x^3 \right) dx^\alpha dx^\beta \qquad (52)$$

that, under a general change of coordinates, transforms to

$$dl^2 = \gamma_{\alpha\beta} \left(x'^1, x'^2, x'^3 \right) dx'^\alpha dx'^\beta \qquad (53)$$

where the functional form $\gamma_{\alpha\beta}$ has to be the same under the homogeneity constraint. Such requirement, in a uniform non Euclidean space, leads to invariance of the three independent differential forms in (36) which are not exact differential of any function of the coordinates and, in tetradic form, are

$$e^{(a)}_\alpha dx^\alpha . \qquad (54)$$

The spatial invariant line element rewrites

$$dl^2 = \eta_{ab} \left(e^{(a)}_\alpha dx^\alpha \right) \left(e^{(b)}_\beta dx^\beta \right) \qquad (55)$$

so that the metric tensor becomes

$$\gamma_{\alpha\beta} = \eta_{ab} e^{(a)}_\alpha e^{(b)}_\beta , \qquad (56)$$

maintaining for η_{ab} the definition (26), as a function of the time variable only.

The symmetry properties for the space determine the specific choice of the basis vectors which, in general, are not orthogonal and consequently the metric η_{ab} is not diagonal.
In pure specific case the relations between the three vectors are

$$e_{(1)} = \frac{1}{v} \left[e^{(2)} \wedge e^{(3)} \right]$$
$$e_{(2)} = \frac{1}{v} \left[e^{(3)} \wedge e^{(1)} \right] \qquad (57)$$
$$e_{(3)} = \frac{1}{v} \left[e^{(1)} \wedge e^{(2)} \right]$$

where v is the product

$$v = \mid e^{(a)}_\alpha \mid = \left(e^{(1)} \cdot \left[e^{(2)} \wedge e^{(3)} \right] \right) \qquad (58)$$

and where it is natural to interpret $\mathbf{e}_{(a)}$ and $\mathbf{e}^{(a)}$ as the Cartesian vectors of components $e^\alpha_{(a)}$ and $e^{(a)}_\alpha$ respectively.

The metric tensor determinant (56) takes the value

$$\gamma = \eta v^2 \qquad (59)$$

being η the determinant of the matrix η_{ab}.

The space invariance as in (53) is equivalent to the invariance of (54), then

$$e^{(a)}_\alpha (x)\, dx^\alpha = e^{(a)}_\alpha (x')\, dx'^\alpha \qquad (60)$$

where $e^{(a)}_\alpha$ on both sides of the equation are the same functions of the old and new coordinates respectively. Making some algebra one gets

$$\frac{\partial x'^\beta}{\partial x^\alpha} = e^\beta_{(a)} (x')\, e^{(a)}_\alpha (x)\,, \qquad (61)$$

which is a system of differential equations to determine the functions $x'^\beta (x)$ on the given basis. The integrability of this system requires the satisfaction of Schwartz's conditions

$$\frac{\partial^2 x'^\beta}{\partial x^\alpha \partial x^\gamma} = \frac{\partial^2 x'^\beta}{\partial x^\gamma \partial x^\alpha}\,, \qquad (62)$$

and by explicit calculations leads to

$$\left[\frac{\partial e^\beta_{(a)} (x')}{\partial x'^\delta} e^\delta_{(b)} (x') - \frac{\partial e^\beta_{(b)} (x')}{\partial x'^\delta} e^\delta_{(a)} (x') \right] e^{(b)}_\gamma (x)\, e^{(a)}_\alpha (x) =$$
$$= e^\beta_{(a)} (x') \left[\frac{\partial e^{(a)}_\gamma (x)}{\partial x^\alpha} - \frac{\partial e^{(a)}_\alpha (x)}{\partial x^\gamma} \right]\,. \qquad (63)$$

Using the properties of the tetradic basis and multiplying both sides of (63) by $e^\alpha_{(d)} (x)\, e^\gamma_{(c)} (x)\, e^{(f)}_\beta (x')$ and some algebra the expression on the left hand side becomes

$$e^{(f)}_\beta (x') \left[\frac{\partial e^\beta_{(d)} (x')}{\partial x'^\delta} e^\delta_{(c)} (x') - \frac{\partial e^\beta_{(c)} (x')}{\partial x'^\delta} e^\delta_{(d)} (x') \right] =$$
$$= e^\beta_{(c)} (x')\, e^\delta_{(d)} (x') \left[\frac{\partial e^{(f)}_\beta (x')}{\partial x'^\delta} - \frac{\partial e^{(f)}_\delta (x')}{\partial x'^\gamma} \right]\,. \qquad (64)$$

Analogously one obtains for the right hand side an identical expression but different only for being function of x. Being the transformation $x \to x'$ arbitrary, both expressions have to be equal to the same constant, giving the group constant of structure[7]

$$\left(\frac{\partial e^{(c)}_\alpha}{\partial x^\beta} - \frac{\partial e^{(c)}_\beta}{\partial x^\alpha} \right) e^\alpha_{(a)} e^\beta_{(b)} = C^c_{\ ab}\,. \qquad (65)$$

---

[7]For the indexes of $C^c_{\ ab}$ the parentheses are unimportant.

Multiplying (65) by $e^\gamma_{(c)}$ gives the uniformity conditions over the space

$$e^\alpha_{(a)} \frac{\partial e^\gamma_{(b)}}{\partial x^\alpha} - e^\beta_{(b)} \frac{\partial e^\gamma_{(a)}}{\partial x^\beta} = C^c_{\ ab} e^\gamma_{(c)}\,. \qquad (66)$$

The expression on the left hand side corresponds to the definition of $\lambda^c_{\ ab}$ (42), then constants.

The antisymmetry property holds, see (47) or (48), with respect to the lower indexes

$$C^c_{\ ab} = -C^c_{\ ba}\,. \qquad (67)$$

Such relations can be rewritten in a compact form in terms of the linear operators

$$X_a = e^\alpha_{(a)} \frac{\partial}{\partial x^\alpha} \qquad (68)$$

so that (66) becomes

$$[X_a, X_b] \equiv X_a X_b - X_b X_a = C^c_{\ ab} X_c\,, \qquad (69)$$

and the homogeneity is expressed as Jacobi identity

$$[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0 \qquad (70)$$

which, in terms of the structure constants, reads

$$C^f_{\ ab} C^d_{\ cf} + C^f_{\ bc} C^d_{\ af} + C^f_{\ ca} C^d_{\ bf} = 0\,. \qquad (71)$$

By dual transformation one can get the two indexes structure constants, more convenient for calculations

$$C^c_{\ ab} = \varepsilon_{abd} C^{dc} \qquad (72)$$

where $\varepsilon_{abc} = \varepsilon^{abc}$ is the Levi-Civita unitary antisymmetric tensor ($\varepsilon_{123} = +1$). The commutation rules (69) expressed in the new constants acquire the compact form

$$\varepsilon^{abc} X_b X_c = C^{ad} X_d\,. \qquad (73)$$

Antisymmetry is implied in the definition (72), while the Jacobi identity (71) becomes

$$\varepsilon_{bcd} C^{cd} C^{ba} = 0 \qquad (74)$$

and (66) for the set (72) is equivalent to the vectorial form

$$C^{ab} = -\frac{1}{v} e^{(a)} \mathrm{rot}\ e^{(b)}\,. \qquad (75)$$

Any linear transformation with constant coefficients

$$e_{(a)} = A^b_a e^{(b)} \qquad (76)$$

shows the non univocal choice of the three reference vectors in the differential forms (54) and with respect to such transformations η_{ab} and C^{ab} behave like tensors. Condition (74) is the only one to be satisfied by the structure

constants, considering only non equivalent combinations with respect to the transformation (76).

The classification of non equivalent homogeneous spaces reduces to the determination of all non equivalent combinations of the constants C^{ab}.

Imposing condition (74) one gets the relations

$$
\begin{aligned}
[X_1, X_2] &= -aX_2 + n_3X_3 \\
[X_2, X_3] &= n_1X_1 \\
[X_3, X_1] &= n_2X_2 + aX_3
\end{aligned} \tag{77}
$$

where a and (n_1, n_2, n_3) are constants related to the structure constants. Upon reduction of all non equivalent sets under rescaling all possible uniform spaces can be summarized following the Bianchi classification as in the following table.

Type	a	n_1	n_2	n_3
I	0	0	0	0
II	0	1	0	0
VII	0	1	1	0
VI	0	1	-1	0
IX	**0**	**1**	**1**	**1**
VIII	**0**	**1**	**1**	**-1**
V	1	0	0	0
IV	1	0	0	1
VII	a	0	1	1
III ($a = 1$) } VI ($a \neq 1$) }	a	0	1	-1

TABLE I. Bianchi classification – Non equivalent structure constants

III. PIECEWISE REPRESENTATION OF THE MIXMASTER

A. Field Equations

In a synchronous reference system, the metric for a homogeneous model writes

$$
ds^2 = dt^2 - \eta_{ab}(t) e_\alpha^{(a)}(x^\gamma) e_\beta^{(b)}(x^\gamma) dx^\alpha dx^\beta \tag{78}
$$

where the reference vectors $e_\alpha^{(a)}$ are determined through (66), once specified the structure constants. The matrix $\eta_{ab}(t)$ describes the temporal evolution of the tridimensional geometry, to be derived from the Einstein's field equations which reduce to an ordinary differential system, involving functions of the t only, provided the projection of all spatial part of vectors and tensors over the tetradic basis chosen, using

$$
\begin{aligned}
R_{(a)(b)} &= R_{\alpha\beta} e_{(a)}^\alpha e_{(b)}^\beta \\
R_{0(a)} &= R_{0\alpha} e_{(a)}^\alpha \\
T_{\alpha\beta} &= T_{(a)(b)} e_\alpha^{(a)} e_\beta^{(b)} \\
T_{\alpha 0} &= T_{(a)0} e_\alpha^{(a)} \\
u^{(a)} &= u^\alpha e_\alpha^{(a)}.
\end{aligned} \tag{79}
$$

Homogeneity reflects over all scalar quantities preventing the presence of any spatial gradient, incompatible with the problem symmetry.

The matrix η_{ab} is the projection over the basis of the spatial metric $\gamma_{\alpha\beta}$ and the role of η_{ab} and η^{ab} to raise and lower the indexes is clear.

The projection of the field equations (21)-(23) over the tetrad gives

$$
R_0^0 = -\frac{1}{2}\dot{\kappa}_{(a)}^{(a)} - \frac{1}{4}\kappa_{(a)}^{(b)}\kappa_{(b)}^{(a)} \tag{80}
$$

$$
R_{(a)}^0 = -\frac{1}{2}\kappa_{(b)}^{(c)}\left(C_{ca}^b - \delta_a^b C_{dc}^d\right) \tag{81}
$$

$$
R_{(b)}^{(a)} = -\frac{1}{2\sqrt{\eta}}\frac{\partial}{\partial t}\left(\sqrt{\eta}\kappa_{(a)}^{(b)}\right) - P_{(b)}^{(a)} \tag{82}
$$

where

$$
\begin{aligned}
\kappa_{(a)(b)} &= \dot{\eta}_{(a)(b)} \\
\kappa_{(a)}^{(b)} &= \dot{\eta}_{(a)(c)}\eta^{(c)(b)}
\end{aligned} \tag{83}
$$

and the dot is the derivative with respect to t and the tridimensional Ricci tensor projected

$$
P_{(a)(b)} = \eta_{(b)(c)} P_{(a)}^{(c)} \tag{84}
$$

in terms of the structure constants becomes

$$
\begin{aligned}
P_{(a)(b)} = -\frac{1}{2}\Big(& C_{\ b}^{cd}C_{cda} + C_{\ b}^{cd}C_{dca} + \\
& -\frac{1}{2}C_b^{\ cd}C_{acd} + C_{\ cd}^c C_{ab}^{\ \ d} + C_{\ cd}^c C_{ba}^{\ \ d}\Big).
\end{aligned} \tag{85}
$$

7

Moreover it is not necessary the explicit form of the vector basis as functions of the coordinate for . This task is related to the symmetry degree of the models considered.

B. Kasner Solution

The simplest homogeneous cosmological model corresponds to the Bianchi type I, whose structure constants identically vanish, implying also the Ricci tensor components to be zero so that

$$\left.\begin{array}{c} e_\alpha^a = \delta_\alpha^a \\ C_{ab}^c \equiv 0 \end{array}\right\} \Longrightarrow P_{ab} = 0 \,. \tag{86}$$

Under this conditions and in empty space, equations (80)-(82) reduce to the system

$$\dot{\kappa}_a^a + \frac{1}{2}\kappa_a^b \kappa_b^a = 0 \tag{87}$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\kappa_a^b\right) = 0 \,. \tag{88}$$

By (88) one gets the first integral

$$\sqrt{\gamma}\kappa_a^b = 2\lambda_a^b = \text{const.} \,. \tag{89}$$

The contraction of the a and b indexes gets

$$\kappa_a^a = \frac{\dot{\gamma}}{\gamma} = \frac{2}{\sqrt{\gamma}}\lambda_a^a \,, \tag{90}$$

and then

$$\gamma = Gt^2 \,, \qquad G = \text{const.} \,. \tag{91}$$

Without loss of generality, upon a coordinates rescaling, put such constant equal to one and then also

$$\lambda_a^a = 1 \,. \tag{92}$$

Substituting (89) in (87) one finds the relation for the constants λ_a^b

$$\lambda_b^a \lambda_a^b = 1 \,. \tag{93}$$

Lowering index b in (89) gives a system of ordinary differential equations with respect to γ_{ab}

$$\dot{\gamma}_{ab} = \frac{2}{t}\lambda_a^c \lambda_{cb} \,. \tag{94}$$

The set of λ_a^c can be considered as a matrix for a linear transformation and by an appropriate change of coordinates (x^1, x^2, x^3) is reducible to a diagonal form. Given p_1, p_2, p_3 as the corresponding matrix eigenvalues, taken real and different, in correspondence of the normalized eigenvectors $\mathbf{n}^{(1)}, \mathbf{n}^{(2)}, \mathbf{n}^{(3)}$, the solution of (94) is

$$\gamma_{ab} = t^{2p_1}n_a^{(1)}n_b^{(1)} + t^{2p_2}n_a^{(2)}n_b^{(2)} + t^{2p_3}n_a^{(3)}n_b^{(3)} \tag{95}$$

where the constant coefficients for the powers of t can be reduced to unity, once rescaled the coordinates. If the tetrad vectors are parallel to the coordinate axes, say (x, y, z), metric reduces to

$$ds^2 = dt^2 - t^{2p_1}dx^2 - t^{2p_2}dy^2 - t^{2p_3}dz^2 \,. \tag{96}$$

Constants p_1, p_2, p_3 are three arbitrary numbers, called Kasner indexes, which have to satisfy the conditions

$$p_1 + p_2 + p_3 = 1 \tag{97}$$
$$p_1^2 + p_2^2 + p_3^2 = 1 \,, \tag{98}$$

as a direct consequence of (94) and (93). Except the cases $(0, 0, 1)$ and $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, Kasner indexes have to be different and one of them can acquire negative value. Given the order

$$p_1 < p_2 < p_3 \,, \tag{99}$$

the corresponding variation interval is

$$-\frac{1}{3} \le p_1 \le 0$$
$$0 \le p_2 \le \frac{2}{3} \tag{100}$$
$$\frac{2}{3} \le p_3 \le 1 \,.$$

These numbers can also be represented in the parametric form

$$p_1(u) = \frac{-u}{1 + u + u^2}$$
$$p_2(u) = \frac{1 + u}{1 + u + u^2} \tag{101}$$
$$p_3(u) = \frac{u(1 + u)}{1 + u + u^2}$$

where the parameter u varies in the interval $1 \le u < +\infty$. When $u < 1$, the parameterization can be reduced to the same variability interval for p_1, p_2, p_3, holding

$$p_1\left(\frac{1}{u}\right) = p_1(u)$$
$$p_2\left(\frac{1}{u}\right) = p_3(u) \tag{102}$$
$$p_3\left(\frac{1}{u}\right) = p_2(u) \,.$$

As functions of u, the parameters $p_1(u)$ and $p_3(u)$ monotonically increase, while $p_2(u)$ monotonically decreases. This metric corresponds to a flat space, even anisotropic, where the volume grows proportionally with increasing t and distances along (y, z) axes increase while along x decrease. The time $t = 0$ is the singular point for the solution and such singularity is not avoidable under

any reference system change, while the invariant quantities of the curvature tensor diverge except the case $p_1 = p_2 = 0$, $p_3 = 1$ in which the metric is reducible to the Galilean form once posed the parameterization

$$t \sinh x^3 = \xi$$
$$t \cosh x^3 = \tau. \tag{103}$$

The metric (96) is the exact solution for the Einstein's equations in empty space but, close the a singular point, say small t, it represents an approximate solution.
A generalized solution coincides only with an approximation of the metric, in the sense that dominant terms of such metric as powers of t have analogous form to (96). In general, given a synchronous reference system, the metric can be expressed in the form (17), where the spatial line element dl is

$$dl^2 = \left(a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta\right) dx^\alpha dx^\beta \tag{104}$$

once posed

$$a = t^{p_l}$$
$$b = t^{p_m} \tag{105}$$
$$c = t^{p_n}$$

and the three tridimensional vectors $\mathbf{l}, \mathbf{m}, \mathbf{n}$ redefine the directions along which the spatial lengths vary following the power laws (105). Such quantities are functions of the spatial coordinates. In view of the fact that exponents (105) are all different, the spatial metric (104) is always anisotropic.
An eventual presence of matter doesn't affect the generality of these conclusions and can be introduced in the metric (104)-(105) via four coordinates functions to determine the initial distribution of matter and the three initial velocity components. The behavior of matter in the vicinity of a singular point is determined by equations of motion in a given gravitational field, following classical hydrodynamics.

C. Oscillatory Regime

Differently from the Kasner solution, in all other homogeneous models the projection of the tridimensional Ricci tensor has components different from zero and the complexification induced prevents an analytical description of the solution, except for the Bianchi type II model. In the following is described the asymptotic behavior of the homogeneous models in the vicinity of a singular point.
The most general and interesting case relies in the Bianchi types VIII and IX models (Mixmaster) [6], while others can be considered as simplifications.
The general solution is, by definition, stable. A perturbation to the system is equivalent to a change in the

initial conditions at a given time, but the general solution satisfies arbitrary initial conditions, then the given perturbation cannot affect the form of the solution.
The structure constants sets considered are (see table)

Bianchi VIII : $\quad C^{11} = C^{22} = 1, C^{33} = -1$
Bianchi IX : $\quad C^{11} = C^{22} = 1 = C^{33} = 1. \tag{106}$

Let's consider the case of Bianchi IX. If the matrix η_{ab} is diagonal, the components $R^0_{(a)}$ of the Ricci tensor in (80)-(82) vanish identically in the synchronous reference system and, by (85), the same holds for the off diagonal components of $P_{(a)(b)}$. The remaining Einstein's equations give for the functions $a(t), b(t), c(t)$ the system

$$\frac{(\dot{a}bc)^\bullet}{abc} = \frac{1}{2a^2b^2c^2}\left[\left(b^2 - c^2\right)^2 - a^4\right]$$
$$\frac{\left(a\dot{b}c\right)^\bullet}{abc} = \frac{1}{2a^2b^2c^2}\left[\left(a^2 - c^2\right)^2 - b^4\right] \tag{107}$$
$$\frac{(ab\dot{c})^\bullet}{abc} = \frac{1}{2a^2b^2c^2}\left[\left(a^2 - b^2\right)^2 - c^4\right]$$

and

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0 \tag{108}$$

depending on the time variable only and $(\,)^\bullet = \frac{d}{dt}$. Equations (107)-(108) are exact and valid also far from the singularity in $t = 0$. Introducing the quantities

$$a = e^\alpha$$
$$b = e^\beta \tag{109}$$
$$c = e^\gamma$$

and the temporal variable τ

$$dt = abc \, d\tau \tag{110}$$

one obtains the simplified form

$$2\alpha_{\tau\tau} = \left(b^2 - c^2\right)^2 - a^4$$
$$2\beta_{\tau\tau} = \left(a^2 - c^2\right)^2 - b^4 \tag{111}$$
$$2\gamma_{\tau\tau} = \left(a^2 - b^2\right)^2 - c^4$$
$$\frac{1}{2}\left(\alpha + \beta + \gamma\right)_{\tau\tau} = \alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau \tag{112}$$

where the index τ refers to the derivative with respect to τ. The system (111), by (112), admits the first integral

$$\alpha_\tau\beta_\tau + \alpha_\tau\gamma_\tau + \beta_\tau\gamma_\tau =$$
$$= \frac{1}{4}\left(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2\right) \tag{113}$$

containing only first derivatives. The Kasner regime (105) is a solution of (111) if all right hand side expressions can be neglected. Such a situation cannot hold

indefinitely (for $t \to 0$), because there is always an increasing term. Suppose the case that, for a certain time interval, some expressions in (111) are negligible (Kasnerian regime): for example, the negative exponent corresponds to the function $a(t)$, so that $p_l = p_1$, and the perturbation to the given regime is due to terms as a^4, meanwhile other terms decrease. Keeping the dominant terms in (111), one reduces to the approximate

$$\alpha_{\tau\tau} = -\frac{1}{2}e^{4\alpha}$$
$$\beta_{\tau\tau} = \gamma_{\tau\tau} = \frac{1}{2}e^{4\alpha}. \tag{114}$$

The solution for those equations describes the evolution of the metric, given the initial condition (105). Without loss of generality, once put

$$p_l = p_1$$
$$p_m = p_2 \tag{115}$$
$$p_n = p_3$$

so that

$$a \sim t^{p_1}$$
$$b \sim t^{p_2} \tag{116}$$
$$c \sim t^{p_3}$$

one obtains

$$abc = \Lambda t$$
$$\tau = \frac{1}{\Lambda}\ln t + \text{const.} \tag{117}$$

where Λ is a constant while initial conditions for (114) are

$$\alpha_\tau = p_1$$
$$\beta_\tau = p_2 \tag{118}$$
$$\gamma_\tau = p_3,$$

and finally

$$\alpha = \Lambda p_1$$
$$\beta = \Lambda p_2 \tag{119}$$
$$\gamma = \Lambda p_3.$$

The first of (114) has the same form as the equation of the unidimensional motion for a point particle in an exponential potential barrier, where α has the role of a coordinate. By this analogy, the initial Kasnerian regime corresponds to a free motion with constant velocity $\alpha_\tau = p_1$. After the bounce on the barrier the particle follows its free motion with the same speed but with opposite sign $\alpha_\tau = -p_1$. Using initial conditions (119), the system (114) integrates as

$$a^2 = \frac{2 \mid p_1 \mid \Lambda}{\cosh(2 \mid p_1 \mid \Lambda\tau)}$$
$$b^2 = b_0{}^2 \exp\left[2\Lambda(p_2 - \mid p_1 \mid)\tau\right]\cosh(2 \mid p_1 \mid \Lambda\tau) \tag{120}$$
$$c^2 = c_0{}^2 \exp\left[2\Lambda(p_3 - \mid p_1 \mid)\tau\right]\cosh(2 \mid p_1 \mid \Lambda\tau)$$

where b_0 and c_0 are integration constants. For the solutions (120), in the limit $\tau \to \infty$, the asymptotic form coincides with (116). For $\tau \to -\infty$, say toward the singularity, the asymptotic forms become

$$a \sim \exp\left[\Lambda p_1 \tau\right]$$
$$b \sim \exp\left[\Lambda(p_2 + 2p_1)\tau\right]$$
$$c \sim \exp\left[\Lambda(p_3 + 2p_1)\tau\right] \tag{121}$$
$$t \sim \exp\left[\Lambda(1 + 2p_1)\tau\right]$$

and writing a, b, c as functions of t one gets again a Kasner behavior, a new Kasner *epoch*,

$$a \sim t^{p_l'}$$
$$b \sim t^{p_m'} \tag{122}$$
$$c \sim t^{p_n'}$$
$$abc = \Lambda' t$$

being

$$p_l' = \frac{\mid p_1 \mid}{1 - 2 \mid p_1 \mid}$$
$$p_m' = -\frac{2 \mid p_1 \mid - p_2}{1 - 2 \mid p_1 \mid} \tag{123}$$
$$p_n' = \frac{p_3 - 2 \mid p_1 \mid}{1 - 2 \mid p_1 \mid},$$

together with

$$\Lambda' = (1 - 2 \mid p_1 \mid)\Lambda. \tag{124}$$

The perturbation causes the transition from a Kasner epoch to another in such a way that the negative power of t passes from the direction **l** to **m**, i.e. if initially p_l is negative, in the new solution one gets $p_m' < 0$. The initial perturbation given by terms such $e^{4\alpha}$ is reduced substantially to zero. For the following evolution it will become the dominant term in another right hand side term and follow a completely analogous analysis.

By the parameterization (101) and the property (102), the passages from a Kasner epoch to another re-express in a suggestive way such that if

$$p_l = p_1(u)$$
$$p_m = p_2(u) \tag{125}$$
$$p_n = p_3(u)$$

then

$$p_l' = p_2(u - 1)$$
$$p_m' = p_1(u - 1) \tag{126}$$
$$p_n' = p_3(u - 1)$$

and is called *BKL map*.

During this transition, the function $a(t)$ gets a maximum value, while $b(t)$ a minima. After that b starts increasing, a decreases and $c(t)$ holds its monotonic decrease.

During the change of Kasner epoch, the bigger of the two positive exponents maintains its character following (126). The following changes of epochs continue until the integer part of the initial value of u becomes smaller then one. After that, following (102) the value of $u < 1$ corresponds to another $u' > 1$.

This reparameterization realizes a new exchange in the two positive indexes among the corresponding directions, say one of the oscillating functions becomes decreasing and vice versa.

The time interval in which one scale factor holds monotonically decreasing while the other two oscillate is sadi *Kasner era*. The sequence of the axis exchange and the order in which two eræof different time duration follows assumes a stochastic character.

D. BKL Map

Each Kasner era holds for an included number of epochs equivalent to the integer part of the corresponding initial value of the parameter u. Being arbitrary and in general irrational, the regimes' alternation continues indefinitely, while for an exact solution the meaning of the exponents p_1, p_2, p_3 loses the role as in the Kasner eræ. Such internal indetermination takes out meaning to the study of privileged sets of values.

The hypothesis underlying this asymptotic regime is based on the possibility to find a time interval during which it is possible to neglect the projection of the tridimensional Ricci tensor.

The presence of matter as a perfect fluid, described by an opportune momentum energy tensor, doesn't affect the characteristics of the regime toward the singularity. Nevertheless the description would get a peculiar evolution for the matter. In such a case, by the hydrodynamic equations, the temporal evolution of the energy density for each Kasner epoch becomes

$$\epsilon \sim t^{-2(1-p_3)} \tag{127}$$

where p_3 is the bigger among the set p_1, p_2, p_3. ϵ monotonically increases with the temporal variable decrement and diverges for $t = 0$, as confirmation of the intrinsic singularity.

In all other Bianchi models the asymptotic regime toward the singularity is overall simplified: the oscillatory behavior disappears because the right hand side members of (111) vanish and the corresponding Kasner epoch is the only one describing the system evolution toward the singularity.

Oscillation amplitudes for α and β are growing during the evolution of an assigned era, independently of the era chosen, provided obviously the oscillating behavior.

In parallel to that, also the Kasner epochs' duration is increasing and the matter density increases monotonically as

$$\frac{\epsilon_0'}{\epsilon_0} \sim A_0{}^{2k} \tag{128}$$

being A_0 the original oscillation amplitude and k the index for the k-th era and it is accelerating in the following one

$$\frac{\epsilon_0''}{\epsilon_0'} \sim A_0'{}^{2k'} \sim A_0{}^{2k^2 k'} \tag{129}$$

and so on, showing how rapid is the density matter growth.

The ongoing series of oscillations accumulate in the vicinity of the singular point. Between any final instant of the universal time t and $t = 0$ there is an infinite number of oscillations. The temporal evolution is given by the natural variable $\ln t$ and not the synchronous time t, going to $-\infty$ toward the singularity.

The study of the iterative properties of the BKL map (126) requires another fundamental property of the parameter u. The s-th corresponds to a succession of decreasing values of the parameter u of the form $u_{max}^{(s)}$ (staring era), $u_{max}^{(s)} - 1, u_{max}^{(s)} - 2, \ldots, u_{min}^{(s)}$. One can distinguish

$$u^{(s)} = k^{(s)} + x^{(s)} \tag{130}$$

with notation

$$\begin{aligned} u_{min}^{(s)} &= x^{(s)} < 1 \\ u_{max}^{(s)} &= k^{(s)} + x^{(s)}, \end{aligned} \tag{131}$$

$u_{max}^{(s)}$ is the maximum value of u, $k^{(s)} = \left[u_{max}^{(s)} \right]$ and[8] $x^{(s)}$ is the fractional part. The succession refers to a decreasing sequence of values of u.

The first Kasner era contains $k^{(s)}$ epochs while the subsequent one, parameterized by

$$\begin{aligned} u_{max}^{(s+1)} &= \frac{1}{x^{(s)}} \\ k^{(s+1)} &= \left[\frac{1}{x^{(s+1)}} \right], \end{aligned} \tag{132}$$

corresponds to the sequence of $k^{(s+1)}$. If the entire arrangement starts from $k^{(0)} + x^{(0)}$, the lengths $k^{(1)}, k^{(2)}, \ldots$ are the numbers involved in the expansion of $x^{(0)}$ in the continuous fraction

$$x^{(0)} = \cfrac{1}{k^{(1)} + \cfrac{1}{k^{(2)} + \frac{1}{k^{(3)} + \ldots}}}, \tag{133}$$

which is finite if corresponds to the expansion of a rational number but in general infinite when the initial value

---

[8] The square brackets denote the integer part function.

11

is irrational.

In any infinite sequence u constructed by (132) there are always arbitrarily small numbers $x^{(s)}$, but different from zero, to which correspond arbitrarily long eræ $k^{(s+1)}$. The value assumed by all terms of such expansion is finite and limited and the set of all $x^{(0)} < 1$ with this property has null measure in the interval $(0, 1)$.

Depending on the strong dependence of the BKL map of the initial conditions the infinite sequence of $\left(k^{(0)}, k^{(1)}, k^{(2)}, \ldots\right)$ assumes a chaotic behavior which requires an appropriate analysis.

E. Statistical Description

Kasner eræ iteration, the specific stochasticity in the parameters sequence and the iterative dynamics require a statistical description.

If the initial conditions don't affect the model evolution, expression (133) is very sensitive to the initial values of u and following the system along many eræ it is possible to determine a specific probability distribution.

Any small change on the initial value of $u^{(0)}$ implies a sequence of numbers k completely different once explicited all the fraction terms, then, approaching the singularity, one gets a stationary probability distribution for the values assumed by the integer part k and by the fractional one x referring to u.

The random nature of the process under which the sequence $k^{(s)}$ acquires asymptotically a stochastic character raises from the transition mechanism between different eræ (132).

Instead of an initial value as in (130) with $s = 0$, let's consider a distribution for $x^{(0)}$ over the range $(0, 1)$, described by the probability distribution $W_0(x)$ for $x^{(0)} = x$ to be in such interval. Given $w_s(x)\,dx$ the probability for the last term of the s-th series $x^{(s)} = x$ to be in the interval dx, the corresponding last term in the set has to be in between $\frac{1}{k+1}$ and $\frac{1}{k}$, necessary for the duration of the s-th series to be k. In this case, the probability for such duration is given by

$$W_s(k) = \int_{\frac{1}{1+k}}^{\frac{1}{k}} w_{s-1}(x)\,dx\,, \qquad (134)$$

and the recurrence formula relating the probability distribution $w_{s+1}(x)$ with $w_s(x)$ is

$$w_{s+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} w_s\left(\frac{1}{k+x}\right). \qquad (135)$$

The recursive definition (135) generates w_{s+n} (n generic integer) which, for increasing s, tends to a stationary probability distribution $w(x)$ where the initial conditions are disappeared

$$w(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} w\left(\frac{1}{k+x}\right), \qquad (136)$$

whose normalized solution of (136) is

$$w(x) = \frac{1}{(1+x)\ln 2}. \qquad (137)$$

Once given the probability distribution for x, the corresponding series length k is derivable as

$$W(k) = \int_{\frac{1}{1+k}}^{\frac{1}{k}} w(x)\,dx = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}. \qquad (138)$$

Finally, being k and x not independent, they have to admit a stationary probability distribution correlated

$$w(k, x) = \frac{1}{(k+x)(k+x+1)\ln 2} \qquad (139)$$

which, given $u = k + x$, rewrites as a stationary distribution for the parameter u

$$w(u) = \frac{1}{u(u+1)\ln 2}. \qquad (140)$$

This is the basic result for the study of the model evolution statistical properties, for the homogeneous cosmological model considered.

The perturbative term of the Kasner regime in the Einstein's equations is identified as the negative power law and the corresponding scale factor. Nevertheless, for $u \gg 1$ in any epoch, the parameterization (101) provides Kasner indexes in the asymptotic form

$$\begin{aligned} p_1 &\cong \frac{1}{u} \\ p_2 &\cong \frac{1}{u} \\ p_3 &\cong 1 - \frac{1}{u^2} \end{aligned} \qquad (141)$$

where p_1, p_2 have absolute values comparable and both close to zero. In this scheme, the perturbation is produced equally by terms such t^{4p_1} and t^{4p_2}. Such situation provides equations whose solution, far from the singularity, has an oscillatory behavior.

The obtained relations for the statistical dynamics during a Kasner era loose their validity when the system acquires a small oscillations regime, due to its strong instability.

IV. COVARIANT APPROACH TO THE MIXMASTER CHAOS

A. The Hamiltonian Formulation

In order to provide a Hamiltonian formulation of the Mixmaster dynamics, we rewrite the line element as follows [11]

$$ds^2 = -N(\eta)^2 d\eta^2 + e^{2\alpha} \left(e^{2\beta}\right)_{ij} \sigma^i \sigma^j \qquad (142)$$

where $N(\eta)$ denotes the lapse function, σ^i are the dual 1-forms associated with the anholonomic basis [9] and β_{ij} is a traceless 3×3 symmetric matrix diag$(\beta_{11}, \beta_{22}, \beta_{33})$; α, N, β_{ij} are functions of η only. Parameterizing the matrix β_{ij} by the usual Misner variables [6]

$$\begin{aligned} \beta_{11} &= \beta_+ + \sqrt{3}\beta_- \\ \beta_{22} &= \beta_+ - \sqrt{3}\beta_- \\ \beta_{33} &= -2\beta_+ \end{aligned} \qquad (143)$$

the dynamics of the Mixmaster model is described by a canonical variational principle

$$\delta I = \delta \int L \, d\eta = 0 \,, \qquad (144)$$

with Lagrangian L

$$L = \frac{6D}{N}\left[-\alpha'^2 + \beta_+'^2 + \beta_-'^2\right] - \frac{N}{D}V\left(\alpha, \beta_+, \beta_-\right) . \qquad (145)$$

Here $()' = \frac{d}{d\eta}$, $D \equiv \det e^{\alpha+\beta_{ij}} = e^{3\alpha}$ and the potential $V(\alpha, \beta_+, \beta_-)$ reads

$$V = \frac{1}{2}\left(D^{4H_1} + D^{4H_2} + D^{4H_3}\right) + \\ - D^{2H_1+2H_2} \pm D^{2H_2+2H_3} \pm D^{2H_3+2H_1} \,, \quad (146)$$

where $(+)$ and $(-)$ refers respectively to Bianchi type VIII and IX, and the anisotropy parameters H_i $(i = 1, 2, 3)$ denote the functions [48]

$$\begin{aligned} H_1 &= \frac{1}{3} + \frac{\beta_+ + \sqrt{3}\beta_-}{3\alpha} \\ H_2 &= \frac{1}{3} + \frac{\beta_+ - \sqrt{3}\beta_-}{3\alpha} \\ H_3 &= \frac{1}{3} - \frac{2\beta_+}{3\alpha} \,. \end{aligned} \qquad (147)$$

In the limit $D \to 0$ the second three terms of the above potential turn out to be negligible with respect to the

---

first one. Let's introduce the new (Misner-Chitré-like) variables

$$\begin{aligned} \alpha &= -e^{f(\tau)}\xi \\ \beta_+ &= e^{f(\tau)}\sqrt{\xi^2 - 1}\cos\theta \\ \beta_- &= e^{f(\tau)}\sqrt{\xi^2 - 1}\sin\theta \,, \end{aligned} \qquad (148)$$

with f denoting a generic functional form of τ, $1 \le \xi < \infty$ and $0 \le \theta < 2\pi$. Then the Lagrangian (145) reads

$$L = \frac{6D}{N}\left[\frac{\left(e^f\xi'\right)^2}{\xi^2 - 1} + \left(e^f\theta'\right)^2\left(\xi^2 - 1\right) - \left(e^f\right)'^2\right] + \\ - \frac{N}{D}V\left(f(\tau), \xi, \theta\right) . \qquad (149)$$

In terms of $f(\tau)$, ξ and θ we have

$$D = exp\left\{-3\xi \cdot e^{f(\tau)}\right\} \qquad (150)$$

and since $D \to 0$ toward the singularity, independently of its particular form, in this limit f must approach infinity. The Lagrangian (145) leads to the conjugate momenta

$$\begin{aligned} p_\tau &= \frac{\partial L}{\partial \tau'} = -\frac{12D}{N}\left(e^f \cdot \frac{df}{d\tau}\right)^2 \tau' \\ p_\xi &= \frac{\partial L}{\partial \xi'} = \frac{12D}{N}\frac{e^{2f}}{\xi^2 - 1}\xi' \\ p_\theta &= \frac{\partial L}{\partial \theta'} = \frac{12D}{N}e^{2f}\left(\xi^2 - 1\right)\theta' \end{aligned} \qquad (151)$$

which by a Legendre transformation make the variational principle (144) assume the Hamiltonian form

$$\delta \int \left(p_\xi\xi' + p_\theta\theta' + p_\tau\tau' - \frac{Ne^{-2f}}{24D}\mathcal{H}\right) d\eta = 0 \,, \quad (152)$$

being

$$\mathcal{H} = -\frac{p_\tau{}^2}{\left(\frac{df}{d\tau}\right)^2} + p_\xi{}^2\left(\xi^2 - 1\right) + \frac{p_\theta{}^2}{\xi^2 - 1} + 24Ve^{2f} . \quad (153)$$

B. Reduced Variational Principle

By variating (152) with respect to N we get the constraint $\mathcal{H} = 0$, which solved provides

$$-p_\tau \equiv \frac{df}{d\tau} \cdot \mathcal{H}_{ADM} = \frac{df}{d\tau} \cdot \sqrt{\varepsilon^2 + 24Ve^{2f}} \qquad (154)$$

where

$$\varepsilon^2 = \left(\xi^2 - 1\right)p_\xi{}^2 + \frac{p_\theta{}^2}{\xi^2 - 1} \qquad (155)$$

in terms of which the variational principle (152) reduces to

---

[9] The dual 1-forms of the considered models are given by:

(Bianchi VIII): $\begin{cases} \sigma^1 = -\sinh\psi\sinh\theta d\phi + \cosh\psi d\theta \\ \sigma^2 = -\cosh\psi\sinh\theta d\phi + \sinh\psi d\theta \\ \sigma^3 = \cosh\theta d\phi + d\psi \end{cases}$

(Bianchi IX): $\begin{cases} \sigma^1 = \sin\psi\sin\theta d\phi + \cos\psi d\theta \\ \sigma^2 = -\cos\psi\sin\theta d\phi + \sin\psi d\theta \\ \sigma^3 = \cos\theta d\phi + d\psi \end{cases}$

$$\delta \int \left(p_\xi \xi' + p_\theta \theta' - f' \mathcal{H}_{ADM} \right) d\eta = 0 \,. \qquad (156)$$

Since the equation for the temporal gauge actually reads

$$N(\eta) = \frac{12D}{\mathcal{H}_{ADM}} e^{2f} \frac{df}{d\tau} \tau' \,, \qquad (157)$$

our analysis remains fully independent of the choice of the time variable until the form of f and τ' is not fixed.

The variational principle (156) provides the Hamiltonian equations for ξ' and θ' [10]

$$\xi' = \frac{f'}{\mathcal{H}_{ADM}} \left(\xi^2 - 1 \right) p_\xi$$

$$\theta' = \frac{f'}{\mathcal{H}_{ADM}} \frac{p_\theta}{(\xi^2 - 1)} \,. \qquad (158)$$

Furthermore can be straightforward derived the important relation

$$\frac{d \left(\mathcal{H}_{ADM} f' \right)}{d\eta} = \frac{\partial \left(\mathcal{H}_{ADM} f' \right)}{\partial \eta} \Longrightarrow$$

$$\Longrightarrow \frac{d \left(\mathcal{H}_{ADM} f' \right)}{df} = \frac{\partial \left(\mathcal{H}_{ADM} f' \right)}{\partial f} \,, \quad (159)$$

i.e. explicitly

$$\frac{\partial \mathcal{H}_{ADM}}{\partial f} = \frac{e^{2f}}{2 \mathcal{H}_{ADM}} 24 \cdot \left(2V + \frac{\partial V}{\partial f} \right) \,. \qquad (160)$$

In this reduced Hamiltonian formulation, the functional $f(\eta)$ plays simply the role of a parametric function of time and actually the anisotropy parameters H_i ($i = 1, 2, 3$) are functions of the variables ξ, θ only

$$H_1 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi} \left(\cos\theta + \sqrt{3} \sin\theta \right)$$

$$H_2 = \frac{1}{3} - \frac{\sqrt{\xi^2 - 1}}{3\xi} \left(\cos\theta - \sqrt{3} \sin\theta \right) \qquad (161)$$

$$H_3 = \frac{1}{3} + 2 \frac{\sqrt{\xi^2 - 1}}{3\xi} \cos\theta \,.$$

Finally, toward the singularity ($D \to 0$ i.e. $f \to \infty$) by the expressions (146, 150, 161), we see that [11]

$$\frac{\partial V}{\partial f} = O \left(e^f V \right) \,. \qquad (162)$$

Since in the domain Γ_H, where all the H_i are simultaneously greater than 0, the potential term $U \equiv e^{2f} V$ can be modeled by the potential walls

---

[10]In this study the corresponding equations for p_ξ' and p_θ' are not relevant.

[11]By $O()$ we mean terms of the same order of the inclosed ones.

$$U_\infty = \Theta_\infty \left(H_1(\xi, \theta) \right) + \Theta_\infty \left(H_2(\xi, \theta) \right) + \Theta_\infty \left(H_3(\xi, \theta) \right) \quad (163)$$

$$\Theta_\infty(x) = \begin{cases} +\infty & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

therefore in Γ_H the ADM Hamiltonian becomes (asymptotically) an integral of motion

$$\forall \{\xi, \theta\} \in \Gamma_H$$

$$\begin{cases} \mathcal{H}_{ADM} = \sqrt{\varepsilon^2 + 24 \cdot U} \cong \varepsilon = E = \text{const.} \\ \dfrac{\partial \mathcal{H}_{ADM}}{\partial f} = \dfrac{\partial E}{\partial f} = 0 \,. \end{cases} \qquad (164)$$

The key point for the use of the Misner-Chitré-like variables relies on the independence of the time variable for the anisotropy parameters H_i.

C. The Jacobi Metric and the Billiard Representation

Since above we have shown that asymptotically to the singularity ($f \to \infty$, i.e. $\alpha \to -\infty$) $d\mathcal{H}_{ADM}/df = 0$ i.e. $\mathcal{H}_{ADM} = \epsilon = E = \text{const.}$, the variational principle (156) reduces to

$$\delta \int \left(p_\xi d\xi + p_\theta d\theta - E df \right) =$$

$$= \delta \int \left(p_\xi d\xi + p_\theta d\theta \right) = 0 \,, \qquad (165)$$

where we dropped the third term in the left hand side since it behaves as an exact differential.

By following the standard Jacobi procedure [26] to reduce our variational principle to a geodesic one, we set $x^{a'} \equiv g^{ab} p_b$, and by the Hamiltonian equation (158) we obtain the metric

$$g^{\xi\xi} = \frac{f'}{E} \left(\xi^2 - 1 \right)$$

$$g^{\theta\theta} = \frac{f'}{E} \frac{1}{\xi^2 - 1} \,. \qquad (166)$$

By these and by the fundamental constraint relation

$$\left(\xi^2 - 1 \right) p_\xi{}^2 + \frac{p_\theta{}^2}{\xi^2 - 1} = E^2 \,, \qquad (167)$$

we get

$$g_{ab} x^{a'} x^{b'} = \frac{f'}{E} \left\{ \left(\xi^2 - 1 \right) p_\xi{}^2 + \frac{p_\theta{}^2}{\xi^2 - 1} \right\} = f' E \,. \quad (168)$$

By the definition $x^{a'} = \frac{dx^a}{ds} \frac{ds}{d\eta} \equiv u^a \frac{ds}{d\eta}$, (168) rewrites

$$g_{ab} u^a u^b \left(\frac{ds}{d\eta} \right)^2 = f' E \,, \qquad (169)$$

14

which leads to the key relation

$$dη = \sqrt{\frac{g_{ab}u^au^b}{f'E}}\ ds\,. \tag{170}$$

Indeed expression (170) together with $p_ξ ξ' + p_θ θ' = Ef'$ allows us to put the variational principle (165) in the geodesic form

$$δ \int f'E\ dη = δ \int \sqrt{g_{ab}u^au^bf'E}\ ds =$$
$$= δ \int \sqrt{G_{ab}u^au^b}\ ds = 0 \tag{171}$$

where the metric $G_{ab} \equiv f'Eg_{ab}$ satisfies the normalization condition $G_{ab}u^au^b = 1$ and therefore [12]

$$\frac{ds}{dη} = Ef' \Rightarrow \frac{ds}{df} = E\,. \tag{172}$$

Summarizing, in the region $Γ_H$ the considered dynamical problem reduces to a geodesic flow on a two dimensional Riemannian manifold described by the line element

$$ds^2 = E^2 \left[\frac{dξ^2}{ξ^2 - 1} + (ξ^2 - 1)\ dθ^2 \right]\,. \tag{173}$$

Now it is easy to check that the above metric has negative curvature, since the associated curvature scalar reads $R = -\frac{2}{E^2}$; therefore the point-universe moves over a negatively curved bidimensional space on which the potential wall (146) cuts the region $Γ_H$. By a way completely independent of the temporal gauge we provided a satisfactory representation of the system as isomorphic to a billiard on a Lobachevsky plane [26].

D. Invariant Lyapunov Exponent

In order to characterize the dynamical instability of the billiard in terms of an invariant treatment (with respect to the choice of the coordinates $ξ$, $θ$), let us introduce the following (orthonormal) tetradic basis

$$v^i = \left(\frac{\sqrt{ξ^2 - 1}}{E}, 0 \right)$$
$$w^i = \left(0, \frac{1}{E\sqrt{ξ^2 - 1}} \right)\,. \tag{174}$$

Indeed the vector v^i is nothing else than the geodesic field, i.e.

---

[12] We take the positive root since we require that the curvilinear coordinate s increases monotonically with increasing value of f, i.e. approaching the initial cosmological singularity.

$$\frac{Dv^i}{ds} = \frac{dv^i}{ds} + Γ^i_{kl}v^kv^l = 0\,, \tag{175}$$

while the vector w^i is parallel transported along the geodesic, according to the equation

$$\frac{Dw^i}{ds} = \frac{dw^i}{ds} + Γ^i_{kl}v^kw^l = 0\,, \tag{176}$$

where by $Γ^i_{kl}$ we denoted the Christoffel symbols constructed by the metric (173). Projecting the geodesic deviation equation along the vector w^i (its component along the geodesic field v^i does not provide any physical information about the system instability), the corresponding connecting vector (tetradic) component Z satisfies the following equivalent equation

$$\frac{d^2Z}{ds^2} = \frac{Z}{E^2}\,. \tag{177}$$

This expression, as a projection on the tetradic basis, is a scalar one and therefore completely independent of the choice of the variables. Its general solution reads

$$Z(s) = c_1 e^{\frac{s}{E}} + c_2 e^{-\frac{s}{E}}\,, \qquad c_{1,2} = \text{const.}\,, \tag{178}$$

and the invariant Lyapunov exponent defined as [16]

$$λ_v = \sup \lim_{s \to \infty} \frac{\ln \left(Z^2 + \left(\frac{dZ}{ds} \right)^2 \right)}{2s}\,, \tag{179}$$

in terms of the form (178) takes the value

$$λ_v = \frac{1}{E} > 0\,. \tag{180}$$

When the point-universe bounces against the potential walls, it is reflected from a geodesic to another one thus making each of them unstable. Though up to the limit of our potential wall approximation, this result shows without any ambiguity that, independently of the choice of the temporal gauge, the Mixmaster dynamics is isomorphic to a well described chaotic system. Equivalently, in terms of the BKL representation, the free geodesic motion corresponds to the evolution during a Kasner epoch and the bounces against the potential walls to the transition between two of them. By itself, the positive Lyapunov number (180) is not enough to ensure the system chaos, since its derivation remains valid for any Bianchi type model; the crucial point is that for the Mixmaster (type VIII and IX) the potential walls reduce the configuration space to a compact region ($Γ_H$), ensuring that the positivity of $λ_v$ implies a real chaotic behavior (i.e. the geodesic motion fills the entire configuration space).

Summarizing, our analysis shows that for any choice of the time variable, we are able to give the above stochastic representation of the Mixmaster model, provided the factorized coordinate transformation in the configuration space

$$\alpha = -e^{f(\tau)}a\left(\theta,\xi\right)$$
$$\beta_+ = e^{f(\tau)}b_+\left(\theta,\xi\right) \qquad (181)$$
$$\beta_- = e^{f(\tau)}b_-\left(\theta,\xi\right),$$

where $f, a, b_\pm$ denote generic functional forms of the variables τ, θ, ξ.

It is worth noting that the success of our analysis, in showing the time gauge independence of the Mixmaster chaos, relies on the use of a standard ADM reduction of the variational principle (which reduces the system by one degree of freedom) and overall because, adopting Misner-Chitré-like variables, the asymptotic potential walls are fixed in time. The difference between our approach and the one presented in [38] (see also for a critical analysis [35]) consists effectively in these features, though in those works is even faced the problem of the Mixmaster chaos covariance with respect to the choice of generic configuration variables.

V. STATISTICAL MECHANICS APPROACH

A. Covariance of the Mixmaster Invariant Measure

In order to reformulate the description of the Mixmaster stochasticity in terms of the Statistical Mechanics [61], we adopt in (156) the restricted time gauge $\tau' = 1$, leading to the variational principle

$$\delta \int \left(p_\xi \frac{d\xi}{df} + p_\theta \frac{d\theta}{df} - \mathcal{H}_{ADM} \right) df = 0. \qquad (182)$$

In spite of this restriction, for any assigned time variable τ (i.e. η) there exists a corresponding function $f(\tau)$ (i.e. a set of Misner-Chitré-like variables) defined by the (invertible) relation

$$\frac{df}{d\tau} = \frac{\mathcal{H}_{ADM}}{12D}N(\tau)e^{-2f}. \qquad (183)$$

As a consequence of the variational principle (182) we have again the expression (160).

In agreement with this scheme, in the region Γ_H where the potential vanishes, we have by (183) $d\mathcal{H}_{ADM}/df = 0$, i.e. $\varepsilon = E = $ const. (by other words the ADM Hamiltonian approaches an integral of motion).

Hence the analysis to derive the invariant measure for the system follows the same lines presented in [48,56].

Indeed we got again a representation of the Mixmaster dynamics in terms of a two-dimensional point-universe moving within closed potential walls and over a negative curved surface (the Lobachevsky plane [48]), described by the line element (173). Due to the bounces against the potential walls and to the instability of the geodesic flow on such a plane, the dynamics acquires a stochastic feature. This system, admitting an "energy-like" constant of motion ($\varepsilon = E$), is well-described by a *microcanonical ensemble*, whose Liouville invariant measure reads

$$d\varrho = A\delta\left(E - \varepsilon\right)d\xi d\theta dp_\xi dp_\theta, \qquad A = \text{const.} \qquad (184)$$

where $\delta\left(x\right)$ denotes the Dirac function. After the natural positions

$$p_\xi = \frac{\varepsilon}{\sqrt{\xi^2 - 1}}\cos\phi, \qquad p_\theta = \varepsilon\sqrt{\xi^2 - 1}\sin\phi, \qquad (185)$$

being $0 \le \phi \le 2\pi$, and the integration over all possible values of ε (depending on the initial conditions, they do not contain any information about the system chaos), we arrive to the uniform invariant measure

$$d\mu = d\xi d\theta d\phi \frac{1}{8\pi^2}. \qquad (186)$$

The validity of our potential approximation is legitimated by implementing the Landau-Raichoudhury theorem [13] near the initial singularity (placed by convention in $T = 0$, where T denotes the synchronous time, i.e. $dT = -N(\tau)d\tau$), we easily get that D vanishes monotonically (i.e. for $T \to 0$ we have $d\ln D/dT > 0$). In terms of the adopted time variable τ ($D \to 0 \Rightarrow f(\tau) \to \infty$), we have

$$\frac{d\ln D}{d\tau} = \frac{d\ln D}{dT}\frac{dT}{d\tau} = -\frac{d\ln D}{dT}N(\tau) \qquad (187)$$

and therefore D vanishes monotonically for increasing τ as soon as, by (183), $d\Gamma/d\tau > 0$.

Now the key point of our analysis is that any stationary solution of the Liouville theorem, like (160), remains valid for any choice of the time variable τ; indeed in [56] the construction of the Liouville theorem with respect to the variables (ξ, θ, ϕ) shows the existence of such properties even for the invariant measure (191).

We conclude remarking how, when approaching the singularity $f \to \infty$ (i.e. $\mathcal{H}_{ADM} \to E$), the time gauge relation (183) simplifies to

$$\frac{df}{d\tau} = \frac{Ee^{-2f+3\xi e^f}}{12}N(\tau)e^{-2f}; \qquad (188)$$

in agreement with the analysis presented in [56], during a free geodesic motion the asymptotic functions $\xi(f), \theta(f), \phi(f)$, are provided by the simple system

$$\frac{d\xi}{df} = \sqrt{\xi^2 - 1}\cos\phi$$
$$\frac{d\theta}{df} = \frac{\sin\phi}{\sqrt{\xi^2 - 1}}$$
$$\frac{d\phi}{df} = -\frac{\xi\sin\phi}{\sqrt{\xi^2 - 1}}. \qquad (189)$$

---

[13]Such a theorem, within the mathematical assumptions founding Einsteinian dynamics, states that in a synchronous reference it always exists a given instant of time in correspondence to which the metric determinant vanishes monotonically.

Once getting $\xi(f)$ as the parametric solution

$$\xi(\phi) = \frac{\rho}{\sin^2 \phi}$$

$$f(\phi) = \frac{1}{2}\text{arctanh}\left(\frac{1}{2}\frac{\sin^2 \phi + a^2\left(1 + \cos^2 \phi\right)}{a\rho\cos\phi}\right) + b$$

$$\rho \equiv \sqrt{a^2 + \sin^2 \phi} \quad a,b = \text{const.} \in \Re \qquad (190)$$

it reduces, for a free geodesic motion, equation (188) to a simple differential one for the function $f(\tau)$.

However, the global behavior of ξ along the whole geodesic flow, is described by the invariant measure (191) and therefore relation (188) takes a stochastic character. If we assign one of the two functions $f(\tau)$ or $N(f)$ analytically, the other one acquires a stochastic behavior. We see how the one-to-one correspondence between any lapse function $N(\eta)$ and the associated set of Misner-Chitrè-like variables, ensures the covariant nature with respect to the time-gauge of the Mixmaster universe stochastic behavior.

B. Quantum Nature of the Mixmaster Chaos

In what follows we will consider the particular case $f(\tau) \equiv \tau$ and to make clear its asymptotic nature, we redefine the invariant measure as follows

$$d\mu = w_\infty(\xi,\theta,\phi)\,d\xi d\theta d\phi \equiv \frac{1}{8\pi^2}d\xi d\theta d\phi . \qquad (191)$$

Summarizing, over the reduced phase space[14] $\{\xi,\theta\} \otimes S^1_\phi$ the distribution w_∞ behaves like the step-function

$$w_\infty(\xi,\theta,\phi) = \begin{cases} \dfrac{1}{8\pi^2} & \forall \; \{\xi,\theta,\phi\} \in \Gamma_H \otimes S^1_\phi \\ 0 & \forall \; \{\xi,\theta,\phi\} \notin \Gamma_H \otimes S^1_\phi \end{cases} . \quad (192)$$

Once performed the transformation (185) over the Hamiltonian equations (158), in the asymptotic limit for which $U \to U_\infty \Rightarrow \varepsilon = E = \text{const.}$, we get in Γ_H the free geodesic motion [56]

$$\frac{d\xi}{d\tau} = \sqrt{\xi^2 - 1}\cos\phi$$

$$\frac{d\theta}{d\tau} = \frac{\sin\phi}{\sqrt{\xi^2 - 1}}$$

$$\frac{d\phi}{d\tau} = -\frac{\xi\sin\phi}{\sqrt{\xi^2 - 1}} . \qquad (193)$$

This dynamical system induces the *stationary* continuity equation for the distribution function $w_\infty(\xi,\theta,\phi)$ describing the ensemble representation

$$\sqrt{\xi^2 - 1}\cos\phi\frac{\partial w_\infty}{\partial \xi} + \frac{\sin\phi}{\sqrt{\xi^2 - 1}}\frac{\partial w_\infty}{\partial \theta} +$$

$$- \frac{\xi\sin\phi}{\sqrt{\xi^2 - 1}}\frac{\partial w_\infty}{\partial \phi} = 0 . \qquad (194)$$

If now we restrict our attention to the distribution function on the configuration space Γ_H

$$\varrho(\xi,\theta) \equiv \int_0^{2\pi} w_\infty(\xi,\theta,\phi)\,d\phi , \qquad (195)$$

by (194) we get for such restricted form the two dimensional continuity equation

$$\sqrt{\xi^2 - 1}\cos\phi\frac{\partial\varrho_\infty}{\partial\xi} + \frac{\sin\phi}{\sqrt{\xi^2 - 1}}\frac{\partial\varrho_\infty}{\partial\theta} = 0 . \qquad (196)$$

The *microcanonical* solution on the whole configuration space $\{\xi,\theta\}$ reads

$$\varrho_\infty(\xi,\theta) = \begin{cases} \dfrac{1}{4\pi} & \forall \; \{\xi,\theta\} \in \Gamma_H \\ 0 & \forall \; \{\xi,\theta\} \notin \Gamma_H \end{cases} . \qquad (197)$$

The main result of [57] and [58,59], is the proof that the chaos of the Bianchi IX model above outlined is an intrinsic feature of its dynamics and not an effect induced by a particular class of references: in fact the whole MCl formalism and its consequences can be developed in a framework free from the choice of a specific time gauge. Since this intrinsic chaos appears close enough to the Big Bang, we infer that it has some relations with the indeterministic quantum dynamics the model performs in the *Planckian era*. This relation between quantum and deterministic chaos is searched in the sense of a semiclassical limit for a canonical quantization of the model [61].

The asymptotical principle (156) describes a two dimensional anholonomic Hamiltonian system, which can be quantized by a natural Schröedinger approach

$$i\hbar\frac{\partial\psi}{\partial\tau} = \hat{\mathcal{H}}_{ADM}\psi , \qquad (198)$$

being $\psi = \psi(\tau,\xi,\theta)$ the wave function for the point-universe and, implementing $\hat{\mathcal{H}}_{ADM}$ (see (164)) to an operator[15], i.e.

$$\xi \to \hat{\xi} , \qquad\qquad \theta \to \hat{\theta} ,$$

$$p_\xi \to \hat{p}_\xi \equiv -i\hbar\frac{\partial}{\partial\xi} , \qquad p_\theta \to \hat{p}_\theta \equiv -i\hbar\frac{\partial}{\partial\theta} , \qquad (199)$$

---

[15]The only non vanishing canonical commutation relations are

$$\left[\hat{\xi},\hat{p}_\xi\right] = i\hbar , \qquad \left[\hat{\theta},\hat{p}_\theta\right] = i\hbar .$$

---

[14]S^1_ϕ denotes the ϕ-circle.

17

the equation (198) rewrites explicitly, in the asymptotic limit $U \to U_\infty$,

$$i\frac{\partial \psi}{\partial \tau} = \sqrt{\hat{\varepsilon}^2 + \frac{U_\infty}{\hbar^2}}\ \psi = \Bigg[-\sqrt{\xi^2-1}\frac{\partial}{\partial \xi}\sqrt{\xi^2-1}\frac{\partial}{\partial \xi} +$$

$$-\frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial \theta}\frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial \theta} + \frac{U_\infty}{\hbar^2}\Bigg]^{1/2}\psi\,, \qquad (200)$$

where we took an appropriate symmetric normal ordering prescription and we left U_∞ to stress that the potential cannot be neglected on the entire configuration space $\{\xi, \theta\}$ and, being infinity out of Γ_H, it requires as boundary condition for ψ to vanish outside the potential walls

$$\psi\left(\partial \Gamma_H\right) = 0\,. \qquad (201)$$

The *quantum* equation (200) is equivalent to the Wheeler-DeWitt one for the same Bianchi model, once separated the positive and negative frequencies solutions [17], with the advantage that now τ is a real time variable[16]. Since the potential walls U_∞ are time independent, a solution of this equation has the form

$$\psi\left(\tau, \xi, \theta\right) = \sum_{n=1}^{\infty} c_n e^{-iE_n\tau/\hbar} \varphi_n\left(\xi, \theta\right) \qquad (202)$$

where c_n are constant coefficients and we assumed a discrete "energy" spectrum because the quantum point-universe is restricted in the finite region Γ_H and the position (202) in (200) leads to the eigenvalue problem

$$\Bigg[-\sqrt{\xi^2-1}\quad \frac{\partial}{\partial \xi}\sqrt{\xi^2-1}\frac{\partial}{\partial \xi} +$$

$$-\frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial \theta}\frac{1}{\sqrt{\xi^2-1}}\frac{\partial}{\partial \theta}\Bigg]\varphi_n =$$

$$= \left(\frac{E_n{}^2 - U_\infty}{\hbar^2}\right)\varphi_n \equiv \frac{E_{\infty n}{}^2}{\hbar^2}\varphi_n\,. \qquad (203)$$

In what follows we search the semiclassical solution of this equation regarding the eigenvalue $E_{\infty n}$ as a finite constant (i.e. we consider the potential walls as finite) and only at the end of the procedure we will take the limit for U_∞ (163).

We infer that in the semiclassical limit when $\hbar \to 0$ and the *occupation number* n tends to infinity (but $n\hbar$ approaches a finite value) the wave function φ_n approaches a function φ as

$$\lim_{\substack{n\to\infty\\ \hbar\to 0}} \varphi_n\left(\xi, \theta\right) = \varphi\left(\xi, \theta\right)\,, \qquad \lim_{\substack{n\to\infty\\ \hbar\to 0}} E_{\infty n} = E_\infty\,. \quad (204)$$

The expression φ is taken as a semiclassical expansion up to the first order, i.e.

---

[16]This equivalence can be easily checked by taking the square of the operators on both sides of the equation.

$$\varphi\left(\xi, \theta\right) = \sqrt{r\left(\xi, \theta\right)}\exp\left\{ i\frac{S\left(\xi, \theta\right)}{\hbar}\right\}\,, \qquad (205)$$

where r and S are functions to be determined. Substituting (205) in (203) and separating the real from the complex part we get two independent equations, i.e.

$$E_\infty{}^2 = \underbrace{\left(\xi^2-1\right)\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{1}{\xi^2-1}\left(\frac{\partial S}{\partial \theta}\right)^2}_{\text{classical term}} +$$

$$-\frac{\hbar^2}{\sqrt{r}}\left[\sqrt{\xi^2-1}\frac{\partial}{\partial \xi}\sqrt{\xi^2-1}\frac{\partial}{\partial \xi} + \frac{1}{\xi^2-1}\frac{\partial^2}{\partial \theta^2}\right]\sqrt{r} \quad (206)$$

where we multiplied both sides by $\hbar^2$ and, respectively,

$$\underbrace{\sqrt{\xi^2-1}\frac{\partial}{\partial \xi}\left(\sqrt{\xi^2-1}\ r\frac{\partial S}{\partial \xi}\right) + \frac{1}{\xi^2-1}\frac{\partial}{\partial \theta}\left(r\frac{\partial S}{\partial \theta}\right)}_{O(1/\hbar)} = 0\,.$$

$$(207)$$

In the limit $\hbar \to 0$ the second term of (206) is negligible meanwhile the first one reduces to the Hamilton-Jacobi equation

$$\left(\xi^2-1\right)\left(\frac{\partial S}{\partial \xi}\right)^2 + \frac{1}{\xi^2-1}\left(\frac{\partial S}{\partial \theta}\right)^2 = E_\infty{}^2\,. \qquad (208)$$

The solution of (208) can be easily checked to be

$$S\left(\xi, \theta\right) = \int \left\{ \frac{1}{\sqrt{\xi^2-1}}\sqrt{E_\infty^2 - \frac{k^2}{\xi^2-1}}\ d\xi + k\ d\theta\right\}\,, \qquad (209)$$

where k is an integration constant.
We observe that (208), through the identifications

$$\frac{\partial S}{\partial \xi} = p_\xi\,, \frac{\partial S}{\partial \theta} = p_\theta \quad \Longleftrightarrow \quad S = \int \left(p_\xi d\xi + p_\theta d\theta\right)\,, \qquad (210)$$

is reduced to the algebraic relation

$$\left(\xi^2-1\right)p_\xi{}^2 + \frac{1}{\xi^2-1}p_\theta{}^2 = E_\infty{}^2\,. \qquad (211)$$

The constraint (211) is nothing more than the asymptotic one $\mathcal{H}_{ADM}^2 = E^2 = \text{const.}$ and can be solved setting

$$\frac{\partial S}{\partial \xi} = p_\xi \equiv \frac{E_\infty}{\sqrt{\xi^2-1}}\cos\phi\,,$$

$$\frac{\partial S}{\partial \theta} = p_\theta \equiv E_\infty\sqrt{\xi^2-1}\sin\phi\,, \qquad (212)$$

where $\phi \in [0, 2\pi[$ is a momentum-function related to ξ and θ by the dynamics, $\varphi(\tau) = \varphi(\xi(\tau), \theta(\tau))$. On the other hand, by (209) we get

$$p_\xi = \frac{1}{\sqrt{\xi^2 - 1}} \sqrt{E_\infty^2 - \frac{k^2}{\xi^2 - 1}} \qquad (213)$$

$$p_\theta = k\,; \qquad (214)$$

to verify the compatibility of these expressions with (212) we use the equations of motion (193) which provide[17]

$$\frac{d\xi}{d\phi} = -\frac{\xi^2}{\xi^2 - 1}\mathrm{ctg}\varphi \Rightarrow \sqrt{\xi^2 - 1}\sin\varphi = c\,, \qquad (215)$$

and c is a constant of integration.

The required compatibility comes from the identification $k = E_\infty c$. Since

$$\lim_{U \to U_\infty} E_\infty = \begin{cases} E & \forall\,\{\xi,\theta\} \in \Gamma_H \\ i\infty & \forall\,\{\xi,\theta\} \notin \Gamma_H \end{cases} \qquad (216)$$

we see by (209) that the solution $\varphi(\xi,\theta)$ vanishes, as due in presence of infinite potential walls, outside Γ_H.

The substitution in (207) of the positions (212) leads to the new equation

$$\sqrt{\xi^2 - 1}\cos\phi\frac{\partial r}{\partial \xi} + \frac{\sin\phi}{\sqrt{\xi^2 - 1}}\frac{\partial r}{\partial \theta} = 0\,. \qquad (217)$$

We emphasize how this equation coincides with (196), provided the identification $r \equiv \varrho_\infty$; it is just this correspondence between the statistical and the semiclassical quantum analysis to ensure that the quantum chaos of the Bianchi IX model approaches its deterministic one in the considered limit.

Any constant function is a solution of (217), but the normalization condition requires $r = 1/4\pi$ and therefore we finally get

$$\lim_{\substack{n \to \infty \\ \hbar \to 0}} |\varphi_n|^2 = |\varphi|^2 \equiv \varrho_\infty = \begin{cases} \dfrac{1}{4\pi} & \forall\ \{\xi,\theta\} \in \Gamma_H \\ 0 & \forall\ \{\xi,\theta\} \notin \Gamma_H \end{cases}\,, \qquad (218)$$

say the limit for the quantum probability distribution as $n \to \infty$ and $\hbar \to 0$ associated to the wave function

$$\psi(\tau,\theta,\xi) = \varphi(\xi,\theta)\,e^{-i\frac{E}{\hbar}} =$$
$$= \sqrt{r}\,\exp\left\{ i\int (p_\xi d\xi + p_\theta d\theta - E_\infty d\tau) \right\} \quad (219)$$

coincides with the classical statistical distribution on the microcanonical ensemble.

Though this formalism of correspondence remains valid for all Bianchi models, only the types VIII and IX admit a normalizable wave function $\varphi(\xi,\theta)$, being confined

---

[17]Indeed, apart from the bounces against the potential walls, (193) describe the whole evolution of the system.

in Γ_H, and a continuity equation (196) which has a real statistical meaning.

Since referred to stationary states $\varphi_n(\xi,\theta)$, the considered semiclassical limit has to be intended in view of a "macroscopic" one and is not related to the temporal evolution of the model [49].

DEDICATION

This paper is dedicated to the memory of Mario Imponente ($\dagger$19 October 2001).

The following references provide a sample of the fundamental literature on the subject presented in this paper and are grouped by year of publication.

---

[1] Bianchi L (1897) *Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti*, in *Opere*, Vol. **9** (Edizione Cremonese, Roma, 1958).

[2] Lyapunov A M (1907). *Ann. Math. Studies* **17**, Princeton, 1947.

[3] Henon M and Heiles (1964). *Astronom.Journ.* **69**, 73.

[4] Anosov D V (1967) *Geodesic Flows on Closed Riemann Manifolds with Negative Curvature (Proc. Steklov Inst. Math* **90** *(1967)*; translation: American Mathematical Society, Providence, 1969 .

[5] Belinski V A and Khalatnikov I M (1969) *Sov. Phys. JETP*, **29**, 911.

[6] Misner C W (1969) *Phys. Rev. Lett.*, **22**, 1071.

[7] Belinski V A, Khalatnikov I M and Lifshitz E M (1970) *Adv. Phys.*, **19**, 525.

[8] Lifshitz E M, Lifshitz I M and Khalatnikov I M (1971) *Sov. Phys. JETP*, **32**, 173.

[9] Chitré D M (1972) *Ph.D. Thesis*, University of Maryland.

[10] Bogoyavlenskii O and Novikov S (1973) *Sov. Phys. JETP*, **37**, 747.

[11] Misner C W, Thorne K S and Wheeler J A, *Gravitation*, edited by W. H. Freeman and Company, New York (1973) p. 812.

[12] Lukash V N, (1974) *Sov. Phys.-JETP*, **40**, n. 5, 792-799.

[13] Grischchuk L P, Doroshkevich A G and Yudin V M (1975) *Sov. Phys.-JETP*, **42**, n.6, 943-949.

[14] Grischchuk L P, Doroshkevich A G e Yudin V M (1975), *Zh. Eksp. Teor. Fiz.* **69** 1857.

[15] Landau L D and Lifshitz E M, (1975) *Classical Theory of Fields*, Fourth Edition, Addison-Wesley, New York.

[16] Pesin Ya B (1977) *UMN (Russian Mathematical Surveys)*, **32**, n.4 55-112

[17] Kuchar K (1981) in Quantum Gravity 2, a Second Oxford Symposium, edited by C.J.Isham, R.Penrose and D.W.Sciama, Clarendon Press Oxford, 1981, 229.

[18] Barrow J D (1982) *Phys. Rep.*, **85**, 1.

[19] Belinski V A, Khalatnikov I M and Lifshitz E M (1982) *Adv. Phys.*, **31**, 639.

[20] Chandrasekhar S, *The Mathematical Theory of Black Holes* (1983).

[21] Chernoff D F and Barrow J D (1983) *Phys. Rev. Lett*, **50**, 134.

[22] Gurzadyan V G and Kocharyan A A (1986) *Preprint EPI-920 (71), Yerevan*, unpublished.

[23] Gurzadyan V G and Kocharyan A A (1987) *Mod. Phys. Lett. A*, **2**, n. 12, 921-927.

[24] Gurzadyan V G and Kocharyan A A (1987) *Sov. Phys.JETP*, **66**, (4) 651.

[25] Gurzadyan V G and Kocharyan A A (1987) *Sov. Phys. Dokl.*, **31**, (4) 275.

[26] Arnold V I (1989) *Mathematical Methods of Classical Mechanics*, Springer-Verlag.

[27] Arnold V I (1989) in :*Dynamical Systems, vol.1, Springer Verlag.*

[28] Berger B K (1991) *Phys. Rev. D*, **49**, 2, 1120-1123.

[29] Berger B K (1991) *Gen. Rel. and Gravit.*, **23**, n. 12, 1385-1402.

[30] Burd A B, Buric N, Tavakol R K (1991) *Class. and Quantum Grav.*, **8**, 123-133.

[31] Pullin J, (1991) *Relativity and Gravitation: Classical and Quantum, Proc. VIIth SILARG Symp. Cocoyoc Mexico 1990*, J C D'Olivio et al. (eds)

[32] Ferraz K, Francisco G and Matsas (1992) *Phys. Lett. A*, **156**, n. 7,8, 407-409.

[33] Ferraz K and Francisco G (1992) *Phys. Rev. D*, **45**, n. 4, 1158.

[34] Giavalisco M and Yahil A (1992) *J. Korean Phys. Soc.*, **25**, S219.

[35] Burd A and Tavakol R (1993) *Phys. Rev. D*, **47**, n. 12, 5336.

[36] Szydlowski M (1993) *Phys. Lett. A* , **176**, 22.

[37] Szydlowski M and Krawiec A (1993) *Phys. Rev. D*, **47**, n. 12, 5323.

[38] Szydlowski M and Lapeta A (1993) *Phys. Lett. A*, **148**, n. 5, 239.

[39] Contopoulos G, Grammaticos B and Ramani A (1994) in: *Deterministic Chaos in General Relativity, Hobill D et al. (eds)*, p. 423.

[40] Berger B K (1994) in: *Deterministic Chaos in General Relativity, Hobill D et al. (eds)*, p.329, World Sci.

[41] Hobill D, Burd A and Coley A, (eds) (1994) *Deterministic Chaos in General Relativity*, World Scientific, Singapore.

[42] Ohanian H C and Ruffini R (1994) *Gravitation and Space Time*, W.W. Norton & Company, New York.

[43] Rugh S E in (1994) *Deterministic Chaos in General Relativity, Hobill D et al. (eds)*, World Scientific, Singapore.

[44] Szydlowski M and Szczesny J, (1994) *Phys. Rev. D* **50**, 819.

[45] Biesiada M (1995) *Class. Quantum Grav.* **12**, 715.

[46] Berger B K, Garfinkle D and Strasser E (1997) *Class. Quant. Grav.* **14** L29-L36.

[47] Cornish N J and Levin J J (1997) *Phys. Rev. Lett.*, **78**, 998; (1997) *Phys. Rev. D*, **55**, 7489.

[48] Kirillov A A and Montani G (1997) *Phys. Rev. D*, **56**, n. 10, 6225.

[49] Kirillov A A and Montani G (1997) *JETP Lett.*, **66**, 7, 475.

[50] Kim S P, (2001) it J. Korean Phys. Soc., **38**, 427.

[51] Kim J B, Kim J H, Lee H K (1998) *J. Korean Phys. Soc.*, **32**, 857.

[52] Kim W T and Yoon M S (1998) *J. Korean Phys. Soc.*, **33**, S495.

[53] Kim J K and Kim S P, (1998) *J. Korean Phys. Soc.*, **33**, 119.

[54] Park M G (1998) *J. Korean Phys. Soc.*, **33**, S581.

[55] Belinski V A (2000) in: *The Chaotic Universe, Gurzadyan V G and Ruffini R (eds)*, p. 350, World Sci.

[56] Montani G (2000) in: *The Chaotic Universe, Gurzadyan V G and Ruffini R (eds)*, 173, World Scientific, Singapore.

[57] Imponente G P, Montani G (2001) *Phys. Rev. D*, **63**, 103501.

[58] Imponente G P, Montani G (2001) available gr-qc/0106028.

[59] Imponente G P, Montani G (2001) in *Similarities and Universalities in Relativistic Flows*, Proceedings of conference SURF 2000, Logos Verlag, Berlin.

[60] Motter A E and Letelier P S *Phys. Lett. A* **285** (2001) 127-13

[61] Imponente G P, Montani G (2002) *Nucl. Phys. B Proc. Suppl.*, **104** 193-196.